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EXACT SOLUTIONS OF A NON-LINEAR FIFTH-ORDER EQUATION FOR DESCRIBING WAVES ON WATER†

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A brief survey of methods for finding exact solutions of non-linear partial differential equations is given. New solutions of a non-linear equation encountered when describing long waves on water are presented.

1. INTRODUCTION

All non-linear differential equations (DEs) can be connectionally divided into three types: exactly solvable, partially solvable and those that have no analytical solution. This classification does not make it possible to give definitions of these classes, which in itself is a difficult problem. For example, even for exactly solvable Hamilton systems (often termed integrable), a distinction is made between exactly solvable DEs in quadratures, entirely integrable systems and complete algebraic integrability [1–3].

The present paper will examine the second of the above types of DE, namely a DE not belonging to the first type but having a certain set of special solutions. It should be noted that all the methods discussed can be used to find exact solutions of DEs of the first type. However, for DEs of the first type there are more general methods of finding analytical solutions than in the case of DEs of the second type, for example, the method of the inverse scattering transform.

Since, after formulating the mathematical model, the question arises of the existence of analytical solutions, the natural requirement for methods of finding these solutions is growing. In recent years a number of studies have appeared that lay claim to new methods for solving non-linear DEs (both ordinary and partial). However, most of them are essentially only certain modifications of the same approach. In our opinion most of the methods proposed for finding analytical solutions can be classified within the framework of a single approach, which has been developed over the past 120 years, starting with the celebrated work of S. V. Kovalevskaya, which dealt with the problem of the motion of a rigid body in a gravitational field.

One of the most remarkable properties that can be possessed by an ordinary differential equation (ODE) is the Painlevé property, which corresponds to the general solution of an ODE without critical movable singular points. Essentially, the presence of this property is a criterion for the existence of a general solution of the ODE.

This property was noted as far back as the nineteenth century in work by Briot and Bouquet [4]. Then Fuchs and Poincaré used this fact to analyse a first-order ODE [4]. However, the first researcher to use the Painlevé property to solve mechanics problems was S. V. Kovalevskaya, who found a new case (different from the cases of Euler and Lagrange) of the exact solution of the problem of the motion of a rigid body in a gravitational field, requiring the general solution to be a meromorphic function [5]. The answer to the question formulated by Kovalevskaya enabled the parameters of the mathematical model to be found for the case of the exact solution. It can be said that Kovalevskaya found the values of the parameters when the system of equations describing the motion of a rigid body has the Painlevé property.

Soon after Kovalevskaya's work, Painlevé attempted to investigate second-order ODEs having the form

$$y_{zz} = R(y,\ y_z,\ z) \tag{1.1}$$

where R is a function that is algebraic with respect to y and y_z and locally analytical with respect to z. The main aim of his investigation was to find all irreducible ODEs of a given type, the general solutions of which have no critical movable singlular points, and to search for ODEs that define new functions.

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Among ODEs of type (1.1), Painlevé and his students found 50 canonical ODEs (a list of them can be found in [6]), the general solutions of which had no critical movable singular points, and here the solutions of 44 of the ODEs of this list were expressed in terms of the solutions of linear ODEs and special functions known at that time. To describe the solutions of the remaining six irreducible ODEs, Painlevé and Hambier introduced new special functions which they called transcendents. The Painlevé transcendents are not classical functions in the usual sense, since they can be defined only as solutions of non-linear second-order ODEs [7, 8] in view of the fact that the solutions of the Painlevé equations have a considerably transcendental dependence on the integration constants.

Interest in the Kovalevskaya–Painlevé methods arose after work by Ablowitz, Ramani and Segur [9, 10], who noted that exactly solvable non-linear partial differential equations (PDEs), using traveling-wave variables, self-similar variables, etc., can be reduced to equations having the Painlevé property. In this context, they suggested a hypothesis concerning the Painlevé property, which can be formulated as follows. If any reduction of a non-linear PDE to an ODE results in an equation having the Painlevé property, then such a non-linear PDE is exactly solvable. Unfortunately, the practical realisation of this hypothesis is inconvenient, and a modification of the application of this hypothesis to non-linear PDEs was proposed in [11].

The essence of this modification is as follows. Suppose the following non-linear DE is given

$$E(u,\ u_x,\ \ldots,\ x,\ t)=0 \tag{1.2}$$

Its solution is sought in the form

$$u=z^{-p}(u_0+u_1z+u_2z^2+\ldots) \tag{1.3}$$

where $z\equiv z(x,t)$ is a new function, $u_j=u_j(z_x,z_t,\ldots)$ are coefficients which depend on the derivatives of the function $z(x,t)$, and p is a number that is determined by equating to zero the expressions with the lowest power of $z(x,t)$ after substituting a series into the DE. The coefficients u_j are found successively by equating to zero the expressions with different powers of the function $z(x,t)$ after substituting a series into the DE.

For an exactly solvable DE, arbitrary coefficients u_r appear in expansion (1.3), the number of which is equal to the order of the DE.

The success of the method was mainly due to the fact that, assuming that $u_j\equiv 0$ for all $j\geqslant p$, for an exactly solvable DE it is possible to construct Bäcklund transformations and Lax pairs [12–15]. However, it turned out that truncated expansions

$$u=u_0z^{-p}+u_1z^{-p+1}+\ldots+u_p \tag{1.4}$$

are also effective in finding special solutions of DEs, which was demonstrated [16–33] in variose modifications of the application of the method of Painlevé expansions to find solutions of a whole series of non-linear DEs. For example, using formula (1.4), special solutions of the generalized Kuramoto–Sivashinsky equation and the Burgers–Korteveg–de Vries equation for specific values of the parameters of these equations were found [23, 24]. The dependence of the coefficients u_j on the derivatives of the function $z(x,t)$ in formula (1.4) was found in this case after substituting (1.4) into the initial DE and subsequently equating to zero expressions with like powers of the function $z(x,t)$. As a result of the substitution, an overdetermined system of DEs with respect to $z(x,t)$ is found, which is converted into an algebraic system of equations if the following is put in it

$$z(x,\ t)=1+c_1\exp(kx-\omega t) \tag{1.5}$$

where c_1, k and ω are constants, which are found from the algebraic system of equations. Since the u_k in Eq. (1.4) depend on the derivatives of the function $z(x,t)$, the coefficients in Eq. (1.4) are transformed in this case into the product of constants multiplied by exponential functions obtained from the differentiation of expression (1.5).

On substituting expression (1.5) into Eq. (1.4), the latter is transformed into a relation consisting of the sum of powers of the expressions

$$r=\frac{a_0\exp(kx-\omega t)}{1+c_1\exp(kx-\omega t)} \tag{1.6}$$

which was used by some researchers to find exact solutions [34, 35]. In a number of cases this sum can be represented in the form of the sum sech$(kx + \omega t)$ and cosech$(kx + \omega t)$, which likewise has often been used to find exact solutions of non-integrable equations [36–38].

It should also be noted that relation (1.6) can be converted to the form

$$r = \frac{\tilde{a}_0}{2}\left[1 + \operatorname{th}\left(\frac{1}{2}(kx - \omega t + b)\right)\right] \tag{1.7}$$

where b is an arbitrary constant. Therefore, the solution of non-linear DEs can also be sought in the form of a power series expansion of hyperbolic tangents [39].

Another of the methods involves finding the solution of a non-linear DE in the form of the sum of a power series of functions that are solutions of the Riccati equation [17, 33] and the equation for an elliptic function [25, 31, 40], and here these expansions are essentially defined in the same way as Painlevé expansions. Thus, most methods for finding exact solutions of non-linear DEs are essentially based on a method directly connected with the Painlevé approach to the analysis of non-linear DEs.

It should be noted that the search for solutions in the form of the sum of hyperbolic tangents, secants and cosecants gives only a certain set of special solutions, generally without exhausting all possible solutions. The full set of special solutions, which has a corresponding mathematical model, can only be found by using the substitution (1.4) and analysing the overdetermined system of equations for $z(x, t)$. This analysis (often fairly complex) enables all the functions to be found for which the initial mathematical model allows of the conventional Painlevé property.

2. SOLUTIONS OF A NON-LINEAR FIFTH-ORDER DIFFERENTIAL EQUATION IN THE FORM OF SOLITARY WAVES

Special solutions of many non-linear PDEs encountered in describing physical effects have now been obtained. One of the few PDEs not studied so far, encountered in describing long waves on water, has the form [4]

$$u_t + u_x + c_1 u u_x + c_2 u_{xxx} + c_3 u_x u_{xx} + c_4 u u_{xxx} + c_5 u_{xxxxx} = 0 \tag{2.1}$$

Special cases of this equation have been analysed in a number of studies. For example, exact solutions of this equation have been found [26] in the form of solitary waves with $c_3 = c_4 = 0$. In the case when $c_3 = c_4 = c_5 = 0$, Eq. (2.1) takes the form of the well-known Korteveg-de Vries equation.

For convenience of calculations, we use the change of variables

$$\begin{gathered} u \to -\frac{60c_2}{c_3 + 2c_4}u, \quad \sqrt{\frac{c_2}{c_5}}(x - t) \to x, \quad \sqrt{\frac{c_2}{c_5}}\frac{c_2^2}{c_5}t \to t \\ a = \frac{60c_4}{c_3 + 2c_4}, \quad b = \frac{60c_1c_5}{c_2(c_3 + 2c_4)}, \quad c_3 \neq -2c_4, \quad c_5 \neq 0 \end{gathered} \tag{2.2}$$

Then Eq. (2.1) is reduced to the form

$$u_t + u_{xxxxx} - a u u_{xxx} - 2(30 - a)u_x u_{xx} - b u u_x + u_{xxx} = 0, \quad ab \neq 0 \tag{2.3}$$

We will show that (2.3) is not an exactly solvable equation. For this purpose, we will apply the Painlevé test to it [11]. The Fuchs indices (i.e. the numbers of arbitrary coefficients in expansion (1.3)) are found as the roots of an equation which depends on the parameter a:

$$(r + 1)(r - 6)(r^3 - 15r^2 + (86 - a)r - 120) = 0 \tag{2.4}$$

We will be interested only in integer solutions of Eq. (2.4), since for non-integer Fuchs indices the equation does not satisfy the requirements of the Painlevé test. Only two values of the parameter a exist for which all Fuchs indices are integers:

$$a = 180: r = -3, -2, -1, 6, 20 \qquad (2.5)$$
$$a = 12: r = -1, 4, 5, 6, 6$$

However, when checking the correspondence of the Fuchs indices to the numbers of arbitrary coefficients of Laurent's expansion (1.3) of the general solution of Eq. (2.3) close to a movable singular point, in both cases the impatibility conditions are unsatisfied. This means that at least one of the arbitrary coefficients of expansion (1.3) is not such, i.e. the general solution of Eq. (2.3) cannot be represented in the form (1.3). Consequently, Eq. (2.3) fails the Painlevé test, i.e. it cannot be classified as an exactly solvable equation.

We will seek possible special solutions by two methods: in the form of a polynomial in functions satisfying the system of Riccati equations [17], and in the form of a polynomial in a function satisfying the non-degenerate elliptic equation [30, 31].

The method in [17] essentially consists of the following: let the functions $\tau(\theta)$ and $\sigma(\theta)$ satisfy the system of Riccati equations

$$\sigma' = -\sigma\tau, \quad \tau' = -\tau^2 - \frac{\mu_0}{K}\sigma + 1 \qquad (2.6)$$

The general solution of system (2.6) has the form

$$\sigma(\theta) = \frac{1}{\mu_0 / K + c_1 \operatorname{ch}\theta + c_2 \operatorname{sh}\theta}, \quad \tau(\sigma) = \frac{c_1 \operatorname{sh}\theta + c_2 \operatorname{ch}\theta}{\mu_0 / K + c_1 \operatorname{ch}\theta + c_2 \operatorname{sh}\theta} \qquad (2.7)$$

Assuming $c_2 = 0$ and $c_1 = 1/K$, we obtain elementary solitary waves (a soliton and a kink respectively)

$$\sigma(\theta) = \frac{K}{\operatorname{ch}\theta + \mu_0}, \quad \tau(\theta) = \frac{\operatorname{sh}\theta}{\operatorname{ch}\theta + \mu_0} \qquad (2.8)$$

System of equations (2.6) admits of the first integral

$$\left(\frac{1}{\sigma} - \frac{\mu_0}{K}\right)^2 - \frac{\tau^2}{\sigma^2} = \frac{1}{K^2} \qquad (2.9)$$

For all values of $\mu_0 = \pm 1$, the functions σ and τ have simple movable poles, and in the case when $\mu_0 = \pm 1$ the function σ has a second-order pole. To eliminate the change in the order of the poles for the function σ, a value $K = \sqrt{\mu_0^2 - 1}$ is selected and system of equations (2.6) is considered with the condition that

$$1 - \tau^2 - 2\mu\sigma + \sigma^2 = 0; \quad \mu = \frac{\mu_0}{\sqrt{\mu_0^2 - 1}}, \quad \mu^2 \neq 1 \qquad (2.10)$$

The solution of Eq. (2.3) is sought in the form of a polynomial in σ and τ, evaluated at the point $\theta = \theta(\xi)$, $\theta' \neq 0$. To simplify subsequent calculations, we will assume that

$$\theta = k\xi, \quad \xi = x - ct, \quad k = \text{const}, \quad c = \text{const} \qquad (2.11)$$

Equation (2.3) admits of a solution in the form of a second-degree polynomial in (σ, τ). Since all powers of τ higher than unity can be eliminated using relation (2.10), the solution of Eq. (2.3) will have the form

$$u = \alpha + \beta\sigma(\theta) + \gamma\tau(\theta) + \lambda\sigma(\theta)\tau(\theta) + \varphi\sigma(\theta)^2 \qquad (2.12)$$

To simplify subsequent calculations, we will assume that all the coefficients are constants.

We substitute expression (2.12) into Eq. (2.3) and then eliminate all the derivatives of (σ, τ) and powers of τ higher than unity by means of relations (2.6) and (2.10).

Equating the coefficients in the expression obtained for different powers of (σ, τ) to zero, we obtain a final system of algebraic equations with unknowns α, β, γ, λ, φ, μ, k and c.

The solving of this system of equations can be divided into two steps at the first step we assume $\mu(\mu^2 - 1) \neq 0$ and obtain solutions of Eq. (2.3) in the form of a combination of waves of type (2.8), and at the second step we assume $\mu = 0$ and find solutions of Eq. (2.3) in the form of a polynomial in sech θ and the θ. All possible solutions of the form (2.12) can be grouped into three families determined by the coefficients with the highest powers of σ and τ:

$$1)\ \varphi=\frac{k^2}{2},\ \lambda=\frac{k^2}{2},\quad 2)\ \varphi=\frac{k^2}{2},\ \lambda=-\frac{k^2}{2},\quad 3)\ \varphi=k^2,\ \lambda=0 \tag{2.13}$$

Note that the solutions of the second family of (2.13) can be obtained from the solutions of the first family of (2.13) by making the replacement $\tau \to -\tau$, and therefore solutions of the second family of (2.13) will not be considered further.

At the first step we find a single solution from the first family of (2.13) that holds for an arbitrary value of the parameter a

$$u = A_0 - \frac{k^2}{2}v,\quad A_0 = \frac{12-b+ak^2}{12a},\quad v = \frac{1+\mu_0 \operatorname{ch}\theta - \sqrt{\mu_0^2-1}\operatorname{sh}\theta}{(\mu_0+\operatorname{ch}\theta)^2} \tag{2.14}$$

This is a two-parameter solution with the parameters k and μ_0, which holds for $\mu_0^2 > 1$. The remaining parameters of solution (2.12) take the following values

$$\beta = -\frac{k^2\mu}{2},\quad \gamma=0,\quad c=\frac{b(b-12)-a(a-12)k^4}{12a} \tag{2.15}$$

Figure 1 shows the solution of (2.14) for $a = -48$, $b = 15$, $k = 0.95$ and $\mu_0 = 2$ (the continuous curve).

At the second step we find two solutions that hold for arbitrary a. The first solution belongs to the first family of (2.13):

$$u = A_0 + \frac{k^2}{2}\,\frac{1\pm\operatorname{ch}\theta}{\operatorname{sh}^2\theta} \tag{2.16}$$

This solution is a special case of solution (2.14) and can be obtained from expression (2.14) by making the replacement $\theta \to \theta + i\pi/2$.

The second solution is from the third family of (2.13)

$$u = A_0 + \frac{k^2}{4} - k^2\operatorname{sech}^2\theta \tag{2.17}$$

This is a one-parameter solution with parameter k. The parameters from (2.12) for this solution take the following values

$$\beta=0,\quad \gamma=0,\quad c=\frac{b(b-12)-16a(a-12)k^4}{12a} \tag{2.18}$$

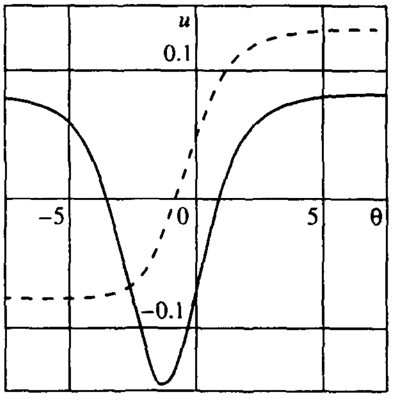

Fig. 1

Solution (2.17) with the replacement $\theta \to \theta + i\pi/2$ is converted into a solution which differs from (2.17) in the replacement $-k^2 \operatorname{sech}^2\theta \to k^2 \operatorname{cosech}^2\theta$.

Note that, when $a = 12$, solutions (2.14), (2.16) and (2.17) correspond to waves propagating at a fixed velocity that is independent of the wave vector.

Note also that, for the parameters of the equation $a = 10$ and $a = -48$, solutions of the form (2.12) exist that differ from those given above.

In the case when $a = 10$ and $\mu(\mu^2 - 1) \neq 0$, and for values of the parameters

$$\alpha = \frac{6 - b + 10k^2}{60}, \quad \beta = -k^2\mu, \quad \gamma = 0$$
$$c = \frac{b(b-6) - 40k^4}{60}, \quad \mu^2 = \frac{2}{3} + \frac{b}{30k^2} \tag{2.19}$$

the solution belonging to the third family of (2.13) can be written in the form

$$u = \frac{6 - b + 10k^2}{60} - \frac{k^2}{1 \pm \varkappa \operatorname{ch}\theta} + \frac{30}{b + 20k^2}\,\frac{k^4}{(1 \pm \varkappa \operatorname{ch}\theta)^2} \tag{2.20}$$

where

$$\varkappa = \sqrt{\frac{b - 10k^2}{b + 20k^2}} \tag{2.21}$$

For imaginary $\varkappa$, solution (2.20), after making the replacement $\theta \to \theta + i\pi/2$, $\varkappa \to i\varkappa$ takes a form which differs from (2.20) by the replacement of $\operatorname{ch}\theta$ by $\operatorname{sh}\theta$ and by a change in the sign of the radicand in Eq. (2.21).

When $a = -48$ and $\mu(\mu^2 - 1) \neq 0$, solutions exist both from the first family of (2.13) and from the third family of (2.13).

The first family of solutions of (2.13) is determined by the following values of the parameters

$$\alpha = \frac{1}{96}(b - 2 - 50k^2), \quad \gamma = \pm\frac{\sqrt{3}}{12}\sqrt{5bk^2 - 348k^4}$$
$$\beta = \gamma - \frac{1}{2}k^2\mu, \quad c = \frac{1}{96}(2b - 21b^2 + 2450bk^2 - 69120k^4) \tag{2.22}$$

and here the solutions in the real domain are only possible for two values of k. The first solution is

$$u = -\frac{1}{48} + \frac{49b}{16704} - \frac{5b}{696}v, \quad k^2 = \frac{5b}{348}, \quad c = \frac{1}{96}\left(2b - \frac{341b^2}{5046}\right) \tag{2.23}$$

The second solution is

$$u = A_1 \pm \frac{b}{144}\,\frac{\sqrt{\mu_0^2 - 1} + \operatorname{sh}\theta}{\mu_0 + \operatorname{ch}\theta} - \frac{b}{144}v, \quad A_1 = -\frac{1}{48} + \frac{11b}{3456}$$
$$k^2 = \frac{b}{72}, \quad c = \frac{1}{96}\left(2b - \frac{11b^2}{36}\right) \tag{2.24}$$

This is a one-parameter solution with parameter μ_0, where $\mu_0^2 > 1$. Solution (2.24) when $b = 15$ and $\mu_0 = 2$ is shown in Fig. 1 by the dashed curve (the plus sign is selected). In (2.23) and (2.24), v is defined by expression (2.14).

Note that solution (2.23) is a special case of solution (2.14), while solution (2.24) when $\mu = 0$ and with the replacement $\theta \to \theta + i\pi/2$ is converted to

$$u = A_1 + \frac{1 \pm \mathrm{ch}\,\theta}{\mathrm{sh}\,\theta}\,\frac{b}{144}\left(\frac{1}{\mathrm{sh}\,\theta} + \varkappa\right), \quad \varkappa = \pm 1 \tag{2.25}$$

The graph of this solution when $b = 75$ and $\varkappa = 1$ is represented in Fig. 2 by the continuous curve (the minus sign is selected).

The group of solutions (2.22) contains a solution with $k^2 = b/48$ that is different from those given above. However, this solution contains an imaginary part and is therefore not considered.

The third family of solutions (2.13) for $a = -48$ and $\mu(\mu^2 - 1) \neq 0$ is determined by the following values of the parameters

$$\alpha = -\frac{1 + 13k^2}{48}, \quad \beta = 0, \quad \gamma = 0$$
$$c = \frac{1}{48}(b + 13bk^2 - 1728k^4), \quad \mu^2 = \frac{29}{35} + \frac{b}{420k^2} \tag{2.26}$$

The one-parameter solution with parameter k form this family has the form

$$u = -\frac{1 + 13k^2}{48} + \frac{420k^4}{b + 348k^2}\,\frac{1}{(1 + \varkappa\,\mathrm{ch}\,\theta)^2} \tag{2.27}$$

where

$$\varkappa = \pm\sqrt{\frac{b - 72k^2}{b + 348k^2}} \tag{2.28}$$

The graph of solution (2.27) with $b = 75$ and $k = 1$ is shown in Fig. 2 by the dashed curve (the plus sign is selected for $\varkappa$).

In the case of imaginary $\varkappa$ in (2.27) and (2.28) after making the replacement $\theta \to \theta + i\pi/2$, $\varkappa \to i\varkappa$, we arrive at a solution that is obtained from (2.27) and (2.28) by replacing $\mathrm{ch}\theta$ by $\mathrm{sh}\theta$ in (2.27) and by changing the sign of the radicand in (2.28).

There is one further solution from the third family of (2.13) for $a = -48$, $\mu(\mu^2 - 1) \neq 0$ and $\gamma = 0$. However, this solution can only be determined implicitly as the solution of the system of four equations

$$-12k^2 + bk^2 + 192k^4 - 576k^2\alpha + 12\beta^2 - 264k^2\beta\mu - 420k^4\mu^2 = 0$$
$$-2\beta + b\beta + 148k^2\beta - 96\alpha\beta + 10k^2\mu + 130k^4\mu + 480k^2\alpha\mu - 60\beta^2\mu - 30k^2\beta\mu^2 = 0$$
$$2ck^2 - 8k^4 - 32k^6 + 2bk^2\alpha - 384k^4\alpha + b\beta^2 + 108k^2\beta^2 + 6k^2\beta\mu + 30k^4\beta\mu + 288k^2\alpha\beta\mu = 0$$
$$c - k^2 - k^4 + b\alpha - 48k^2\alpha = 0 \tag{2.29}$$

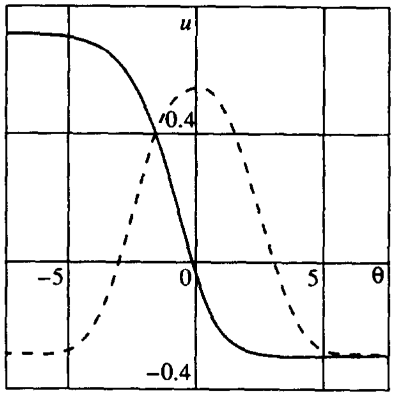

Fig. 2

When $a = -48$ and $\mu = 0$, the third family of solutions (2.13) is determined by the following values of the parameters

$$\alpha = \frac{1}{96}(b-2-200k^2), \quad \beta = 0, \quad \gamma = \pm\frac{\sqrt{5bk^2-1392k^4}}{2\sqrt{3}}$$
$$c = \frac{1}{96}(2b-21b^2+9800bk^2-1105920k^4) \tag{2.30}$$

One of the solutions belonging to this family has the form ($k^2 = b/288$)

$$u = A_1 \pm \frac{b}{144}\,\text{th}\,\theta - \frac{b}{288}\,\text{sech}^2\,\theta \tag{2.31}$$

The family of solutions that is determined by the values of parameters (2.30) includes, besides (2.31), a solution with $k^2 = 5b/1392$ (a special case of solution (2.17)) and a solution with $k^2 = b/192$ with an irremovable imaginary part.

Solution (2.31) with the replacement $\theta \to \theta + i\pi/2$ is converted into a solution which differs from (2.31) by the replacement of thθ and sechθ by cthθ and cosechθ, respectively.

When $a = -48$ and $\mu = 0$, the third family of solutions (2.13) is determined by the following values of the parameters

$$\alpha = \frac{1}{96}(b-2+148k^2), \quad \beta = \pm\frac{\sqrt{5bk^2+696k^4}}{2\sqrt{3}}, \quad \gamma = 0$$
$$c = \frac{1}{96}(2b-21b^2-4900bk^2-270720k^4) \tag{2.32}$$

This family of solutions includes a solution with $k^2 = -5b/696$ (a special case of solution (2.17)), a solution with $k^2 = A_4^+$, $c = c^+$ and $b < 0$, which, after making the replacement $\theta \to \theta + i\pi/2$, reduces to the form

$$u = A_2^+ \pm A_3^+\,\text{cosech}\,\theta + A_4^+\,\text{cosech}^2\,\theta \tag{2.33}$$

and also a solution with $k^2 = A_4^-$, $c = c^-$ and $b < 0$

$$u = A_2^- \pm A_3^-\,\text{sech}\,\theta - A_4^-\,\text{sech}^2\,\theta \tag{2.34}$$

Here

$$A_2^{\pm} = -\frac{1158+(161\pm 37\sqrt{14})b}{55584}, \quad A_3^{\pm} = \frac{\sqrt{1355\sqrt{14}\pm 4712}}{2316}b$$
$$A_4^{\pm} = -\frac{20\pm\sqrt{14}}{2316}b, \quad c^{\pm} = \frac{223494+(46853\pm 10825\sqrt{14})b}{10727712}b \tag{2.35}$$

The graph of solution (2.34) with $b = -100$ is shown in Fig. 3 by the continuous curve (the minus sign is selected).

3. SOLUTIONS OF A NON-LINEAR FIFTH-ORDER DIFFERENTIAL EQUATION IN THE FORM OF KNOIDAL WAVES

To find a solution of Eq. (2.3) in the form of a knoidal wave, the method described in earlier papers [30, 31] is used.

Integrating Eq. (2.3), written in travelling-wave variables, we obtain

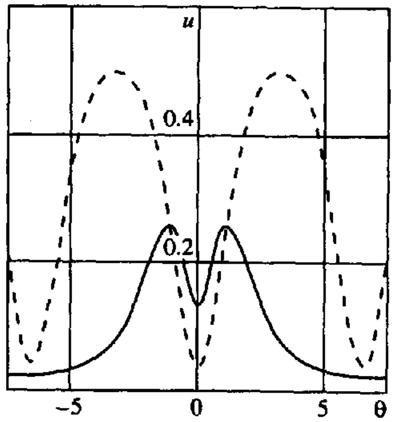

Fig. 3

$$u_{xxxx} - auu_{xx} + u_{xx} + \frac{3}{2}(a-20)u_x^2 - \frac{b}{2}u^2 - cu + d = 0 \tag{3.1}$$

where d is the integration constant.

The general solution of Eq. (3.1) has a second-order pole, and therefore we will seek a solution in the form

$$u = \alpha R(x) + \beta \tag{3.2}$$

where α and β are constants, and R is the solution of the equation

$$R_x^2 = 4R^3 + fR^2 + gR + h \tag{3.3}$$

Substituting expression (3.2) into Eq. (3.1) and equating to zero coefficients of different powers of $R(x)$, we obtain the following values of the parameters of the equation

$$\alpha = 1, \quad \beta = \frac{12-b+af}{12a}, \quad g = \frac{(12-b)b + a(12c + (a-12)f^2)}{12a(a-12)}$$

$$h = \frac{1}{432a^2(a-12)^2}[2b^3(a-6) - 36b(48 - 4a(1+2c-3f) + a^2(c-f))$$
$$+b^2(288 + 36a(f-1) - 3a^2f) + a(a-12)(36c(8+af) + a(-288d + f^3(a-12)))]$$

where it is assumed that $a \neq 12$. In the case when $a = 12$, we find

$$\alpha = 1, \quad \beta = \frac{12-b+12f}{144}, \quad c = \frac{b(b-12)}{144}$$
$$g = -\frac{144b - 24b^2 + b^3 + 41472d - 144bf^2}{1728b}$$

Let $R_1 < R_2 < R_3$ be real and different roots of the equation corresponding to the vanishing of the right-hand side of Eq. (3.3). Then, Eq. (3.1) has a solution in the form of a knoidal wave

$$u = \frac{12-b+af}{12a} + R_2 - (R_2 - R_1)\,\mathrm{cn}^2\left(\sqrt{R_3 - R_1}\,x;\ \sqrt{\frac{R_2 - R_1}{R_3 - R_1}}\right) \tag{3.4}$$

The graph of this solution when $a = -48$, $b = -100$, $f = -8$, $R_1 = 0.9$, $R_2 = 1.36$ and $R_3 = 1.44$ is represented in Fig. 3 by the dashed curve.

Consider the limiting case $R_2 = R_3$. Then the solution of (3.4) takes the form

$$u = \frac{12 - b + af}{12a} + R_1 + (R_2 - R_1)\,\mathrm{th}^2\sqrt{R_2 - R_1}\,x \tag{3.5}$$

where R_1 and R_2 are defined in terms of the arbitrary parameters f and c. A solution of the form (3.5) has already been obtained. It is identical with (2.17).

Thus, special solutions (2.14) and (2.17) of Eq. (2.3) have been found that hold for any values of a. For two values of a, additional special solutions have been obtained: (2.20) for $a = 10$ and (2.24), (2.27), (2.31), (2.33) and (2.34) for $a = -48$. A solution (3.4) of the knoidal wave type has also been found.

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